Journal of Civil Engineering and Environmental Technology
p-ISSN: 2349-8404; e-ISSN: 2349-879X; Volume-7, Issue-2; April-June, 2020 pp. 108-111

http://www.krishisanskriti.org/Publication.html

# Mathematical Investigation on How Increasing Greenhouse Gases in Atmosphere Affects Surface Temperature of Earth

**Nandish Khandhar**

*Class 12th, Oberoi International School, Mumbai, India*

**Abstract—*The report explores how the percentage composition of greenhouse gases in the air affects the surface temperature of the earth. This is done using mathematical greenhouse models that model the incoming and outgoing radiation on the earth from the sun. The focus will be on using a 3-layer mathematical model to model the atmosphere and surface of the planet, which will help in finding out the average surface temperature of the planet. This will be used to model the temperature changes of the atmosphere and the results will be graphed to predict temperature changes in the past century.***

***Keywords: Greenhouse; Mathematical model; Surface temperature***

## 1. INTRODUCTION

In this generation, world governments focus mainly on economic growth and environmental sustainability remains largely disregarded. A direct consequence of this is that the $CO_2$ concentration has increased almost 30% in the past 100 years, and global warming is giving rise to not only melting of glaciers, but also life to new bacteria and viruses that were previously dormant due to low temperatures. In this situation, sustainability of resources and the planet itself should be the primary concern.In this paper, the 3-layer model, along with data for increasing concentration of greenhouse gases will be used in order to predict the actual average temperature of the earth and its trends.This paper uses concepts such as albedo, emissivity and Stephan-Boltzmann's constant to calculate the power emitted by a body. **Albedo**$(\alpha)$ is a measure of the amount of radiation that has hit the surface has been reflected without being absorbed. The earth has an albedo of $\boldsymbol{\alpha = 0.3}$. Secondly, emissivity $(\varepsilon)$ is a concept that describes how bodies emit radiation. A theoretically perfect "blackbody" is a perfect absorber and radiator of energy, with no reflected radiation. It has emissivity of 1, but all other bodies are known as grey bodies, and radiate only a fraction of what a black body of the same temperature would radiate. Emissivity is therefore the ratio of emitted radiation by a grey body compared to energy emitted by a black body of same temperature. The earth's atmosphere has an emissivity of $\mathbf{0.61}$**,** but in this paper it is assumed the earth's surface to be a black body with emissivity of **1.**Lastly, the **Solar Constant** $(K_s)$for earth is the amount of power received by the earth per unit area on a theoretical surface perpendicular to the Sun's rays at mean distance from the Sun. It is fairly constant, but changes every year by a small percentage depending on where is the earth is in its orbit. The accepted value is $K_s = 1366\ Wm^{-2}$.

## 2. CREATING THE ATMOSPHERIC MODEL

This 3-layer model assumes that earth has one surface layer along with 2 layers of atmosphere with different temperatures.

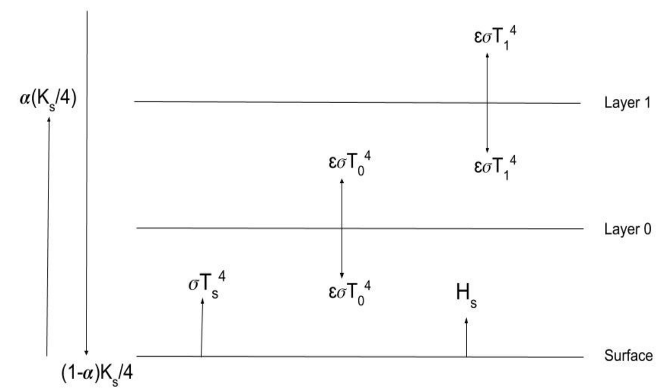


**Fig. 1. Emissions from each layer**

To create this model, the amount of radiation emitted by each layer needs to be found out. The average power per unit area received by the earth from the sun is the solar constant divided by 4, due to the different angles at which the light hits the earth due to its spherical shape. The earth reflects 30% of this radiation and thus absorbs 70%, given by the equation: $(1-\alpha)\,{}^{K_s}/_4$. The surface then radiates back this energy. Stephan's Law says that the power radiated by a blackbody is proportional to it's Kelvin temperature to the fourth power. Adding a proportionality constant, the expression becomes:

$P = \sigma {T_s}^4$, where $T_s$ is the surface temperature and $\sigma$ is the Stephan-Boltzmann's constant: $5.67 \cdot 10^{-8}\ Wm^{-2}K^{-4}$. Layer 0 of the atmosphere absorbs a part of the radiation emitted by the surface $(\varepsilon\sigma {T_s}^4)$ and the rest, $(1-\varepsilon)\sigma {T_s}^4$ is reflected upwards. This is because, according to Kirchhoff's Law of thermal radiation, **emissivity equals absorptivity**.The atmosphere emits the absorbed radiation in two directions, up and down, leading to the total emitted radiation by layer 0 = $2\epsilon\sigma T_0{}^4$. The radiation that was reflected upwards, ($(1-\varepsilon)\sigma T_s{}^4$) and the one half of the radiation emitted by layer 0, ($\epsilon\sigma T_0{}^4$) travels to layer 1, where some of it is absorbed by the layer, $\varepsilon((1-\varepsilon)\sigma T_s{}^4 + (\epsilon\sigma T_0{}^4))$ and the rest is radiated into space. Similar to layer 0, layer 1 also emits radiation in 2 directions, given by the equation: $2\epsilon\sigma T_1{}^4$.At equilibrium, the total incoming radiation, $I_{in}$ is equal to the total outgoing radiation, $I_{out}$. The **first equation**arises by equating incoming and outgoing radiation, as shown in Figure 1:

$$(1-\alpha)\frac{K_s}{4} = (1-\varepsilon)^2\sigma T_s{}^4 + (1-\varepsilon)\varepsilon\sigma T_0{}^4 + \varepsilon\sigma T_1{}^4$$

To complete the model, 2 more equations need to be found, since there are 3 variables in the equation. Since the goal is to find how the surface temperature varies with time, temperatures $T_0$ and $T_1$ need to be written in terms of $T_s$.

**Layer 0:** For this layer of the atmosphere, at constant temperature, incoming radiation equals outgoing radiation. It receives $\varepsilon\sigma T_s{}^4 + H_s$ from the surface, where $H_s$ is the heat transferred in the form of conduction and convection due to evaporation of water from the oceans. This layer also receives one half of the radiation emitted by layer 1 ($\varepsilon \times \varepsilon\sigma T_1{}^4 = \varepsilon^2\sigma T_1{}^4$). The **second** equation is:

$$\varepsilon^2\sigma T_1{}^4 + \varepsilon\sigma T_s{}^4 + H_s = 2\varepsilon\sigma T_0{}^4$$

**Layer 1:** This layer receives one half of the radiation emitted by layer 0, ($\varepsilon^2\sigma T_0{}^4$) and the part of the radiation emitted by the surface that was reflected upwards by layer 0, ($\varepsilon(1-\varepsilon)\sigma T_s{}^4$). The **third** equation becomes:

$$(\varepsilon(1-\varepsilon)\sigma T_s{}^4 + \varepsilon^2\sigma T_0{}^4 = 2\varepsilon\sigma T_1{}^4$$

These three equations can be solved simultaneously and an expression for $I_{net} = I_{in} - I_{out}$ can be found. Taking $T_1{}^4 = \frac{(\varepsilon(1-\varepsilon)\sigma T_s{}^4 + \varepsilon^2\sigma T_0{}^4}{2\varepsilon\sigma}$ from equation 3 and substituting into equation for layer 0 gives the expression:

$$\varepsilon^2\sigma\frac{(\varepsilon(1-\varepsilon)\sigma T_s{}^4 + \varepsilon^2\sigma T_0{}^4}{2\varepsilon\sigma} + \varepsilon\sigma T_s{}^4 + H_s = 2\varepsilon\sigma T_0{}^4$$

The value of $H_s$,according to NASA is 25.5% of the power per unit area absorbed by the earth. Taking 25.5% of $(1-\alpha)\frac{K_s}{4} = 0.255 \times 239.05$,the final value of $H_s$ is approximately $61Wm^{-2}$. With the help of this value, this equation can be manipulated in order to obtain an expression for $T_0{}^4$. Substituting the values of the constants and rearranging the equation gives the value for $T_0{}^4$ to be:

$$T_0{}^4 = T_s{}^4 + 1509769094$$

To find the expression for $T_1{}^4$ can be found by substituting the value for $T_0{}^4$ in equation for either Layer 0 or 1:

$$(\varepsilon(1-\varepsilon)\sigma T_s{}^4 + \varepsilon^2\sigma(T_s{}^4 + 1509769094) = 2\varepsilon\sigma T_1{}^4$$

The final expression after rearranging the equation is:

$$T_1{}^4 = 0.38T_s{}^4 + 296730462.5$$

Substituting these expressions into equation 1 gives:

$$(1-\alpha)\frac{K_s}{4} = (1-\varepsilon)^2\sigma T_s{}^4 + (1-\varepsilon)\varepsilon\sigma(T_s{}^4 + 1509769094) + \varepsilon\sigma(0.38T_s{}^4 + 296730462.5)$$

The LHS = RHS when the net radiation is 0, but otherwise the equation for net radiation after expanding, rearranging and taking the RHS to the left side is:

$$I_{net} = 214.7 - 3.14 \times 10^{-8}T_s{}^4$$

This equation now needs to be used in order to find the change in temperature of the earth each year so that a graph can be made with time on the x-axis and surface temperature on the y-axis. For this, an equation relating temperature and net radiation needs to be found.

Surface heat capacity ($C_s$) is the amount of energy needed to raise the temperature of a unit area of a planets surface by one degree Kelvin. It can be written as:

$$C_s = \frac{Q}{A\Delta T}$$

Taking energy (Q) as power multiplied by time and then taking the power per unit area as intensity ($I_{net}$), the expression for $\Delta T$ is:

$$\Delta T = \frac{I_{net} \times \Delta t}{C_s}$$

Where $\Delta t$ is the time in years and $C_s$ is a constant that has the value of $16.9Jm^{-2}K^{-1}$ for earth. $\Delta t$ can be taken as one year since net radiation will be calculated per year.

$$\Delta T = \frac{214.7 - 3.14 \times 10^{-8}T_s{}^4}{16.9}$$

For this report, the initial/base temperature will be taken from the year 1900 since that is the time the activities ofthe industrial revolution started to have a considerable effect on the temperature of the earth. The initial temperature of the year

1900 according to NASA is $286.85K$ or 13.7℃. For a sample calculation, the $\Delta T$ in the year 1900 would be:

$$\Delta T = \frac{214.7 - 3.14 \times 10^{-8} \times 286.85^4}{16.9} = 0.1246\ ℃/K$$

$$Temperature\ in\ 1901 = 286.85K + 0.1246 = 286.97$$

Using Microsoft Excel, the following data was calculated and the $\Delta T$ for each year was found.

**TABLE 1: I-net and temperature from 1900 to 1905**

| Year | $I_{net}(Wm^{-2})$ | Temperature(℃) |
|---|---|---|
| 1900 | 2.1069 | 13.70 |
| 1901 | 1.7371 | 13.82 |
| 1902 | 1.4318 | 13.93 |
| 1903 | 1.1799 | 14.01 |
| 1904 | 0.9722 | 14.08 |
| 1905 | 0.8009 | 14.14 |

This is a small part of the the table that extends from 1900 to 2018 (2019 and 2020 were not used due to the drop in industrial activity because of the Coronavirus pandemic). The graph plot for time vs temperature is given below:

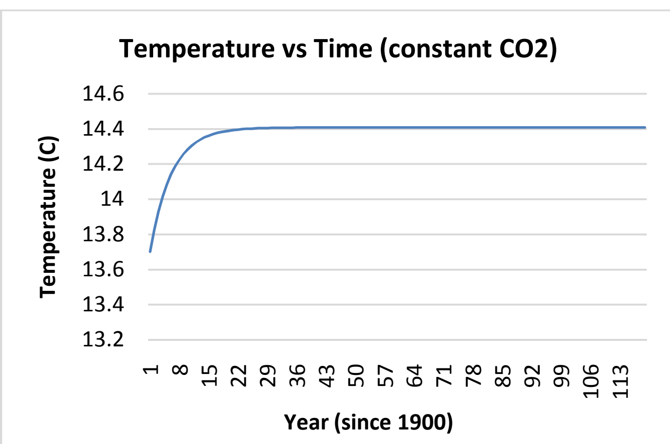


**Fig. 2. Graph of temperature vs time (1900 onwards)**

The graph indicates that after a certain point in time, with constant amount of greenhouse gases in the atmosphere, the net radiation becomes 0 and the $\Delta T$ also becomes stable. The temperature becomes constant at 14.4℃ and does not increase. This can be improved on by taking into consideration the effect of increasing concentration of the main greenhouse gas, $CO_2$. It accounts for 76% of greenhouse gas emissions due to human activity.

## 3. THE GREENHOUSE EFFECT

The greenhouse effect occurs due to the nature of covalent bonds in the $CO_2$ molecule.Even though the molecule itself is non polar, the C=O bond is polar due to the electronegativity difference between carbon and oxygen. The natural vibration of the molecule is equal to the wavelength of Infrared radiation. When the IR radiation strikes the molecule, the bonds stretch asymmetrically, causing a change in the bond length.

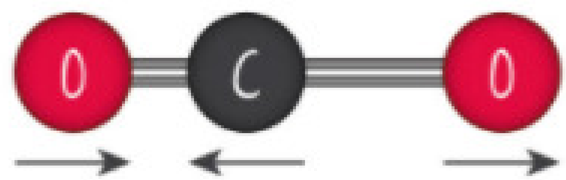


**Fig. 3. Asymmetric stretching of bonds**

This causes a change in the dipole moment of the molecule, and that is the reason it absorbs the IR radiation that was emitted by the surface. This causes the greenhouse effect, leading to higher surface temperatures.The formula of radiative forcing that gives the effect of increasing $CO_2$ concentration on the temperature of the earth is as follows:

$$\Delta T = \lambda \times 5.35 \ln \frac{C}{C_o}$$

where $C_o$ is the base concentration in parts per million (ppm) and $\lambda$ is a climate sensitivity parameter, which has a constant value of 0.8 for earth. Using $CO_2$ concentration data from the government agency "National Oceanic and Atmospheric Administration" or "NOAA" the $\Delta T$ due to the greenhouse effect can be calculated. The base concentration of the year 1900 has been taken as 294.22 ppm. The $\Delta T$ calculated due to the greenhouse effect will be added to the total temperature calculated in Figure 2. This means that as $CO_2$ concentration increases, the global average surface temperature will keep increasing. The table below shows the $CO_2$concentrations along with thecalculated$\Delta T$ and the final temperatures due to the greenhouse effect for the first 5 years.

**TABLE 2: CO2 levels and temperature from 1900 to 1905**

| **Year** | **$CO_2$level (ppm)** | **$\Delta T$ (K)** | **Temperature (℃)** |
|---|---|---|---|
| 1900 | 294.22 | 0.010 | 13.70 |
| 1901 | 294.91 | 0.020 | 13.83 |
| 1902 | 295.61 | 0.023 | 13.95 |
| 1903 | 295.8 | 0.026 | 14.03 |
| 1904 | 295.99 | 0.069 | 14.11 |
| 1905 | 299.02 | 0.072 | 14.21 |

The full table extends from 1900 to 2018. The graph of temperature against time with greenhouse effect is shown below:

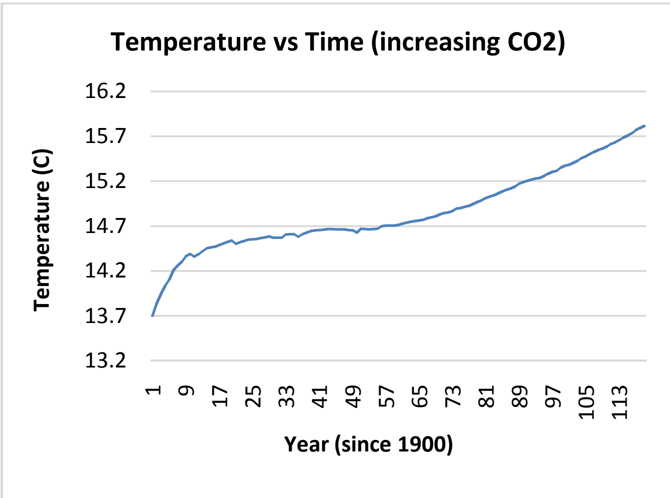


**Fig. 4. Graph of temperature vs time (1900 onwards)**

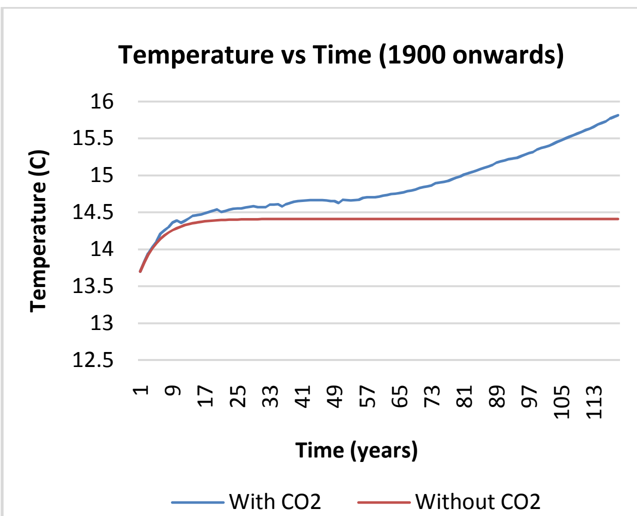


**Fig. 5. Both graphs on one axis (with and without CO2)**

The above graph proves that with the increase in carbon dioxide levels, the average surface temperature increases. The graph below shows both the graphs on one axis:

## 4. EVALUATION OF RESULTS

The results show that the final average global temperature in 2018 with increasing greenhouse gas concentration taken into consideration is 15.81 ℃ or $288.96K$. According to the NOAA and NASA, the 2018 average global temperature was $15.6 \pm 0.13$ ℃. This means that the value found from the model has an error of 0.13% and is relatively accurate.

## 5. CONCLUSION

To conclude, global warming and human activity is severely damaging the environment. Earth's temperatures are rising faster than ever and will continue to at this rate of industrialization. The error caused in the final value of the temperature may be caused due to not taking into consideration different greenhouse gases and their effects. For example, $CH_4$ is also a greenhouse gas that was not explored in this report. Considering the atmosphere as 2 distinct layers may also be a limitation to the report that can be explored in further research. Scientists and researchers have been debating on how to solve the problem of increasing temperatures on earth, and this has led to many solutions like sustainable energy from nuclear fusion, to solar panels etc... Future generations will have to create sustainable technology and reduce the carbon footprint in order to reduce the greenhouse effect.

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